QUANTUM IMMANANTS
AND HIGHER CAPELLI IDENTITIES

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ABSTRACT. We consider remarkable central elements of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(n))$ which we call quantum immanants. We express them in terms of generators E_{ij} of $\mathcal{U}(\mathfrak{gl}(n))$ and as differential operators on the space of matrices. These expressions are a direct generalization of the classical Capelli identities. They result in many nontrivial properties of quantum immanants.

1. INTRODUCTION

1.1. By E_{ij} denote the standard generators of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(n))$. Consider the following element of $\mathcal{U}(\mathfrak{gl}(n))$

$$(1.1) \quad \mathbf{C} = \sum_{s \in S(n)} \operatorname{sgn}(s) \, E_{1,s(1)}(E_{2,s(2)} + \delta_{2,s(2)}) \ldots (E_{n,s(n)} + (n-1)\delta_{n,s(n)}) \, .$$

Symbolically we can write

$$(1.1') \qquad \mathbf{C} = \text{row-det} \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} + 1 & & E_{2n} \\ \vdots & & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} + n - 1 \end{bmatrix} \, ,$$

where the row-determinant of this matrix with non-commutative entries is defined by (1).

Denote by $\mathrm{M}(n)$ the space of $n \times n$-matrices. Denote by $\Bbbk$ the ground field. We suppose that $\operatorname{char} \Bbbk = 0$. Consider the representation L of $\mathcal{U}(\mathfrak{gl}(n))$ in the space $\Bbbk[\mathrm{M}(n)]$, $i, j = 1, \ldots, n$ given on the generators by the following formula

$$(1.2) \qquad\qquad L(E_{ij}) = \sum_{\alpha} x_{i\alpha} \partial_{j\alpha} \, .$$

It is well known that L maps $\mathcal{U}(\mathfrak{gl}(n))$ isomorphically onto the algebra of all differential operators with polynomial coefficients on the space $\mathrm{M}(n)$ that commute with

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the right action of $GL(n)$. Introduce formal matrices E, X, D with (i,j)-th entry equal to E_{ij}, x_{ij}, ∂_{ij} respectively. Then (2) can be written as

$$(1.3) \qquad\qquad L(E) = X \cdot D',$$

where prime means transposition. The celebrated Capelli identity [C] asserts that

$$(1.4) \qquad\qquad L(\mathbf{C}) = \det X \cdot \det D.$$

Here $\det X$ and $\det D$ are ordinary determinants. Observe that the RHS of (4) visibly commutes with the left action of $GL(n)$ so that $\mathbf{C}$ is in fact a central element of $\mathcal{U}(\mathfrak{gl}(n))$. The Capelli identity is one of the most imporatant results of classical invariant theory [H]. Modern approaches to this identity were developed in [HU], [KS] and by other authors. (See, for example, references in the cited papers.) One of these modern approaches is based on the notion of a quantum determinant for Yangian $Y(\mathfrak{gl}(n))$ (see [MNO]). There are a q-analog of the Capelli identity [NUW] and its super analog [N].

In this paper we study some remarkable generalizations of the Capelli element which we call quantum immanants. In some sence we replace the determinant in $(1')$ and (4) by the trace of a arbitrary polynomial representation of $GL(n)$. Our approach is based on R-matrix formalism (however we do not consider Yangians).*

Normally the Capelli element (1) is defined by following column determinant

$$\mathbf{C} = \sum_{s \in S(n)} \operatorname{sgn}(s)\,(E_{s(1),1} + (n-1)\delta_{s(1),1})(E_{s(2),2} + (n-2)\delta_{s(2),2})\ldots E_{s(n),n}.$$

Quantum immanants (see below) can be also rewritten in the column form.

1.2. Introduce the formal matrix

$$E(u) = \left[E_{ij} - u \cdot \delta_{ij}\right]_{i,j=1}^{n}.$$

Here u is a formal variable. A formal $n \times n$ matrix A with entries a_{ij} from a noncommutative algebra $\mathbb{A}$ can be considered as an element

$$A = \sum_{ij} a_{ij} \otimes e_{ij} \ \in\ \mathbb{A} \otimes \mathrm{M}(n),$$

where e_{ij} are standard matrix units in $\mathrm{M}(n)$. The tensor product of two such matrices A and B is defined by

$$A \otimes B = \sum_{i,j,k,l} a_{ij} b_{kl} \otimes e_{ij} \otimes e_{kl} \ \in \mathbb{A} \otimes \mathrm{M}(n)^{\otimes 2}.$$

In the space $(\mathbb{k}^n)^{\otimes k}$ acts the symmetric group $S(k)$ so that we have a representation

$$\mathbb{k}[S(k)] \to \mathrm{M}(n)^{\otimes k}.$$

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*Recently the author proved a more general Capelli-type identity which involves not only the center but the whole algebra $\mathcal{U}(\mathfrak{gl}(n))$. The proof (which does not require R-matrices) will be

It can be shown (in fact this is a way to prove (4); see [MNO] and below) that

(1.5) $$\mathbf{C} = (n!)^{-1}\,\mathrm{tr}\left(E \otimes E(-1) \otimes \cdots \otimes E(-n+1)\cdot \mathrm{Alt}\right),$$

where Alt is the anti-symmetrizer

$$\mathrm{Alt} = \sum_{s\in S(n)} \mathrm{sgn}(s)\cdot s \in \Bbbk[S(n)],$$

and the trace of an element of $\mathbb{A} \otimes \mathrm{M}(n)^{\otimes n}$ is defined by

$$\mathrm{tr}\left(\sum_{i_1,j_1,\ldots,i_n,j_n} a_{i_1,j_1,\ldots,i_n,j_n} \otimes e_{i_1,j_1} \otimes \cdots \otimes e_{i_n,j_n}\right) = \sum_{i_1\ldots,i_n} a_{i_1,i_1,\ldots,i_n,i_n} \in \mathbb{A}.$$

In (5) the algebra $\mathbb{A}$ is $\mathcal{U}(\mathfrak{gl}(n))$. In customary notations (5) can be rewritten as

$$\mathbf{C} = (n!)^{-1} \sum_{i_1,\ldots,i_n} \sum_{s\in S(n)} \mathrm{sgn}(s)\, E_{i_1,i_{s(1)}} \cdots \left(E_{i_n,i_{s(n)}} + (n-1)\delta_{i_n,i_{s(n)}}\right).$$

It is easy to see that the RHS of (4) can be written in a similar form

$$\det X \cdot \det D = (n!)^{-1}\,\mathrm{tr}\left(X^{\otimes n}\cdot (D')^{\otimes n}\cdot \mathrm{Alt}\right).$$

Since the representation L is faithful let us omit the letter L and identify elements of $\mathcal{U}(\mathfrak{gl}(n))$ with differential operators. Then the Capelli identity can be restated as follows:

(1.6) $$\mathrm{tr}\left(E \otimes E(-1) \otimes \cdots \otimes E(-n+1)\cdot \mathrm{Alt}\right) = \mathrm{tr}\left(X^{\otimes n}\cdot (D')^{\otimes n}\cdot \mathrm{Alt}\right).$$

The identity (6) is true also for the action of $GL(n)$ on rectangular $n \times m$ matrices. In this case the matrices X and D are also rectangular $n \times m$ matrices. From now on we consider this general rectangular case.

1.3. Now we can formulate higher Capelli identities. Let μ be a partition such that $\ell(\mu) \leq n$. Put $k = |\mu|$. Let χ^μ be the character of the group $S(k)$ corresponding to the partition μ

$$\chi^\mu = \sum_{s\in S(k)} \chi^\mu(s)\cdot s \in \Bbbk[S(k)].$$

Let T be a standard tableau of shape μ. Let ξ_T be the corresponding vector of the Young orthonormal basis. Consider the matrix element

$$\psi_T = \sum_{s\in S(k)} (s\cdot \xi_T, \xi_T)\cdot s \in \Bbbk[S(k)].$$

The element

$$P_T = \frac{\dim \mu}{k!}\,\psi_T \in \Bbbk[S(k)]$$

acts in as orthogonal projection onto ξ_T. in the irreducible $S(k)$-module corresponding to μ and as zero operator in other irreducible $S(k)$-modules.

Suppose $\alpha = (i,j)$ is a cell of μ. The number $c(\alpha) = j - i$ is called the *content* of the cell α. Suppose α is the l-th cell in the tableau T. Put

$$c_T(l) = j - i.$$

For example, if

$$T = \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$$

THEOREM (*Higher Capelli identities*). *For all partitions μ, $\ell(\mu) \le n$ and any standard tableau T of shape μ*

$$(1.7) \qquad \mathrm{tr}\left(E \otimes E(c_T(2)) \otimes \cdots \otimes E(c_T(k)) \cdot P_T \right) = \mathrm{tr}\left(X^{\otimes k} \cdot (D')^{\otimes k} \cdot \chi^\mu/k! \right).$$

In particular the LHS of (7) *does not depend on the choice of T.*

If $\mu = (1^n)$ then (7) turns into (6). Below in (3.24') we shall obtain another version of the identity (7) which turns into the original definition of the Capelli identity if $\mu = (1^n)$. The particular cases $\mu = (1^k)$, $k = 1, \ldots, n$ of this theorem are also known as Capelli identities [HU]. A different approach to the analogs of Capelli identities for $\mu = (k)$ can be found in [N].

EXAMPLE. Suppose $n = 1$ and $\mu = (k)$. Then (7) reads as follows

$$x\frac{d}{dx}\left(x\frac{d}{dx} - 1 \right) \ldots \left(x\frac{d}{dx} - k + 1 \right) = x^k \frac{d^k}{dx^k}.$$

This identity can be easily verified by induction.

1.4. Consider the highest terms of the both sides of (7) with respect to the natural filtration in $\mathcal{U}(\mathfrak{gl}(n))$. A simple calculation shows that

$$\mathrm{tr}(E^{\otimes k} \cdot s) = \mathrm{tr}(E^{\otimes k} \cdot tst^{-1}) + \text{ lower terms}$$

for all $s, t \in S(k)$. Next observe that

$$\sum_{t \in S(k)} t\, P_T\, t^{-1} = \chi^\mu.$$

Therefore the LHS of (7) equals

$$\mathrm{tr}(E^{\otimes k} \cdot \chi^\mu/k!) + \text{ lower terms.}$$

Since X and D commute modulo lower terms the highest terms of the LHS and the RHS of (7) agree by virtue of (3). The structure of this highest term is similar to the definition of the Schur function via characteristic map [M1]. Suppose $g \in GL(n)$. It follows from the classical decomposition of $(\mathbb{k}^n)^{\otimes k}$ as a $GL(n) \times S(k)$-module that the function

$$(1.8) \qquad \mathrm{tr}\left(g^{\otimes k} \cdot \chi^\mu/k! \right)$$

is equal to the trace of g in the irreducible $GL(n)$-module with highest weight μ or, in other words, to the Schur polynomial in eigenvalues of g. Denote the polynomial (8) in matrix elements x_{ij} of by $s_\mu(X)$

$$(1.9) \qquad s_\mu(X) = \mathrm{tr}\left(X^{\otimes k} \cdot \chi^\mu/k! \right).$$

Given a matrix $A = [a_{ij}]$, $i, j = 1, \ldots, k$, the number

$$\sum_{s \in S(k)} \chi^\mu(s)\, a_{1,s(1)} \ldots a_{k,s(k)}$$

is called the *μ-immanant* of the matrix A. If $\mu = (1^k), (k)$ then the μ-immanant turns into determinant and permanent respectively. Note that (9) is the sum of μ-immanants of principal k-submatrices (with repeated rows and columns) of the

1.5. I wish to thank V. Ginzburg, S. Kerov, S. Khoroshkin and M. Noumi for helpful discussions. I am especially grateful to M. Nazarov; this paper would be hardly possible without numerios discussion with him. They helped me very much with the proof of (2.2) (see also paragraph 2.3 below).

Quantum immanats and higher Capelli identities arose from our joint work with Olshanski [OO]. The discussions we had with G. Olshanski during the work on [OO] were very useful for me. His critical comments concerning this text were also very useful.

I have to mention that the structure LHS of (7) is very close to the fusion process from [KuS], [KuSR], [KuR] and [Ch].

2. Quantum immanants and s^*-functions.

2.1. Denote the LHS of (1.7) by $\mathbb{S}_\mu$:

$$(2.1) \qquad \mathbb{S}_\mu = \mathrm{tr}\left(E \otimes E(c_T(2)) \otimes \cdots \otimes E(c_T(k)) \cdot P_T\right) \in \mathcal{U}(\mathfrak{gl}(n)).$$

Below we shall see that this definition does not depend on the choice of T. Because of the structure of the highest term of (1) and by analogy to the quantum determinant let us call this element the *quantum μ-immanant*. We shall see that quantum immanants have many remarkable properties.

2.2. In the next section we prove that the quantum immanant $\mathbb{S}_\mu$ lies in the center $\mathfrak{Z}(\mathfrak{gl}(n))$ of the algebra $\mathcal{U}(\mathfrak{gl}(n))$. Next we calculate the eigenvalue of $\pi_\lambda(\mathbb{S}_\mu)$ where π_λ is the irreducible representation of $\mathcal{U}(\mathfrak{gl}(n))$ with highest weight λ. We shall prove that

$$(2.2) \qquad \pi_\lambda(\mathbb{S}_\mu) = s_\mu^*(\lambda),$$

where s_μ^* is the *shifted Schur polynomial* (see [OO]). The short name of them is s^*-*polynomials*.

The definition of s^*-polynomials is the following. Put

$$(a \downarrow b) = a(a-1)\ldots(a-b+1)$$
$$(a \uparrow b) = a(a+1)\ldots(a+b-1).$$

These products are called falling and raising *factorial powers*. Put also

$$\delta = (n-1,\ldots,1,0)$$

By definition [OO]

$$(2.3) \qquad s_\mu^*(x_1,\ldots,x_n) = \frac{\det\left[(x_i + \delta_i \downarrow \mu_j + \delta_j)\right]}{\det\left[(x_i + \delta_i \downarrow \delta_j)\right]}.$$

Observe that the denominator in (3) equals the Vandermode determinant in variables $x_i + \delta_i$. Observe also that the numerator in (3) is a skew-symmetric function in $x_i + \delta_i$ and hence is divisible by the denominator.

These polynomials were proposed by G. Olshanski in [Ol2]. They differ by shift of variables from the *factorial* Schur polynomials which were studied by L. C. Bieden-

A. Hamel [GH], I. Goulden and C. Greene [GG], and I. G. Macdonald [M2]. The shift of variables is essential. For example, in contrast to factorial Schur polynomials the s^*-polynomials are stable in the following sence

$$(2.4) \qquad\qquad s_\mu^*(x_1, \ldots, x_n, 0) = s_\mu^*(x_1, \ldots, x_n).$$

This stability allows to introduce s^*-*functions* in countable many variables variables as in [M1]. For example the number $s_\mu^*(\lambda)$, where λ is a partition, is well defined.

The s^*-functions have a lot of interesting properties. Their detailed exposition can be found in [OO]. Some of them have a natural interpretation in terms of quantum immanants.

2.3. Denote the differential operator in the RHS of (1.7) by Δ_μ. We prove in the next section that $\Delta_\mu \in \mathfrak{Z}(\mathfrak{gl}(n))$. (Recall that we identify $\mathcal{U}(\mathfrak{gl}(n))$ with the algebra of the right invariant differential operators.) The higher Capelli identities (1.7) will be proved in two steps: first we prove (2) and then

$$(2.6) \qquad\qquad \pi_\lambda(\Delta_\mu) = s_\mu^*(\lambda).$$

The proof of (6) is much more simple than the proof of (2).

As mentioned in the introduction the discussions with M. Nazarov were very helpful for me during the proof of (2). In particular, M. Nazarov drew my attenton to the importance of (3.8). He also conjectured that the eigenvalue in π_λ of the central element (3.43) is equal to $s_\mu^*(\lambda)$. Recently he has found a new proof of (2).

2.4. The proof of (6) will be based on the two following properties of the s^*-functions that are very simple and very useful at the same time. The two theorems we prove below will be also used in the forthcoming paper by A. Molev and M. Nazarov concerning Capelli-type identities for other classical groups. Many other applications of them can be found in [OO].

The following vanishing and characterization theorems is a way to control lower terms of inhomogeneous polynomials s_μ^*. In particular cases similar argument was used by many people (see, for example, [HU]). In the context of Capelli identities it was developed in full generality by S. Sahi in [S]. In this paper S. Sahi considered polynomials that satisfy (2.7–8) and have a more general symmetry than the shifted symmetry. He found an inductive formula for them. However, this formula is very complicated. In our situation we have much more simple formulas.

Denote by $H(\mu)$ the product of the hook lenghts of all cells of μ

$$H(\mu) = \prod_{\alpha \in \mu} h(\alpha).$$

Let λ be another partition $\lambda_1 \geq \lambda_2 \geq \ldots$. Write $\mu \subset \lambda$ if $\mu_i \leq \lambda_i$ for all i. We have

VANISHING THEOREM.

$$(2.7) \qquad\qquad s_\mu^*(\lambda) = 0 \quad \text{unless} \quad \mu \subset \lambda,$$
$$(2.8) \qquad\qquad s_\mu^*(\mu) = H(\mu).$$

Proof. Observe that

Suppose $\lambda_l < \mu_l$ for some l. Then in the matrix

$$\left[(\lambda_i + n - i \mid \mu_j + n - j)\right]$$

all entries with $i \geq l$ and $j \leq l$ vanish. Hence its determinant vanishes. Since the denominator in (3) does not vanish (7) follows.

Next in the matrix

$$\left[(\mu_i + n - i \mid \mu_j + n - j)\right]$$

all entries with $i > j$ vahish. Hence its determinant equals

$$\prod_i (\mu_i + n - i)! \,.$$

Therefore

(2.9) $$s_\mu^*(\mu) = \prod_i (\mu_i + n - i)! \,/\, \prod_{i<j} (\mu_i - \mu_j - i + j) \,.$$

Recall that there are two formulas for the dimension of the irreducible representation of the symmetric group labeled by μ

(2.10) $$\dim \mu = |\mu|! \,/\, H(\mu)$$

(2.11) $$= |\mu|! \prod_{i<j} (\mu_i - \mu_j - i + j) \,/\, \prod_i (\mu_i + n - i)! \,.$$

Thus (9) equals $H(\mu)$. $\quad\square$

2.5. By $\Lambda^*(n)$ denote the algebra of polynomials in variables $x_1, \ldots, x_n$ which are symmetric in new variables $x_1 + \delta_1, \ldots, x_n + \delta_n$. We call such polynomials *shifted symmetric* [OO]. It is clear that $s_\mu^* \in \Lambda^*(n)$.

Observe that the highest term of any polynomial from $\Lambda^*(n)$ is a symmetric polynomial. It is easy to see that the shifted Schur polynomials s_μ^*, $\ell(\mu) \leq n$ form a linear basis in $\Lambda^*(n)$. We have

CHARACTERIZATION THEOREM. *Any of the two following properties determines the polinomial $s_\mu^* \in \Lambda^*(n)$ uniquely:*

(A) $\deg s_\mu^* \leq |\mu|$ *and*

$$s_\mu^*(\lambda) = \delta_{\mu\lambda} H(\mu)$$

for all λ such that $|\lambda| \leq |\mu|$;

(B) *the highest term of s_μ^* is the ordinary Schur function s_μ and*

$$s_\mu^*(\lambda) = 0$$

for all λ such that $|\lambda| < |\mu|$.

Proof. Prove part (A). We have to prove that

$$\left.\begin{array}{l} f \in \Lambda^*(n)\,, \\ \deg f \leq |\mu|\,, \\ f(\lambda) \quad 0 \ \text{for all} \ \lambda \ |\lambda| \leq |\mu|\ \ell(\lambda) \leq n \end{array}\right\} \implies f = 0 \,.$$

Put $k = |\mu|$. The polynomials $\{s^*_\lambda\}, |\lambda| \le k, \ell(\lambda) \le n$, is a linear basis in subspace of $\Lambda^*(n)$ which consists of polynomials of degree $\le k$. Hence

$$(2.12) \qquad f = \sum c_\lambda s^*_\lambda, \quad |\lambda| \le k, \ell(\lambda) \le n,$$

for some coefficients c_λ. Show that $c_\lambda = 0$ for all λ. Suppose $c_\nu \ne 0$ for some ν. Choose the partition ν so that $c_\nu \ne 0$ and $c_\eta = 0$ for all η, $|\eta| < |\nu|$. Evaluate (12) at ν. By the vanishing theorem we obtain

$$0 = c_\nu H(\nu) \,.$$

Thus $c_\nu = 0$.

Prove part (B). Suppose there are two such elements f_1 and f_2 of Λ^*. Then $\deg(f_1 - f_2) < |\mu|$ and $(f_1 - f_2)(\lambda) = 0$ for all λ such that $|\lambda| < |\mu|$. By part (A) we have $f_1 - f_2 = 0$. $\square$

2.6. By (2) the vanishing and characterization theorems can be restated in terms of quantum immanants. Sometimes it is convenient to use the following normalized quantum immanants $\overline{\mathbb{S}}_\mu$. Put

$$(2.14) \qquad (a \upharpoonright \mu) = \prod_{\alpha \in \mu} (a + c(\alpha)) \,,$$

where $c(\alpha)$ is the content of the cell $\alpha \in \mu$. This is a generalization of the factorial powers. We have $(a \upharpoonright (k)) = (a \upharpoonright k)$ and $(a \upharpoonright (1^k)) = (a \downharpoonright k)$. Next put

$$(2.15) \qquad \overline{\mathbb{S}}_\mu = \frac{1}{(n \upharpoonright \mu)} \mathbb{S}_\mu \quad \in \mathfrak{Z}(\mathfrak{gl}(n)) \,.$$

By virtue of (2), vanishing theorem, and the well known formula for the dimension of the representation π_λ

$$(2.16) \qquad \dim_{GL(n)} \lambda = \frac{(n \upharpoonright \mu)}{H(\mu)} \,,$$

we have

$$(2.17) \qquad \operatorname{tr} \pi_\lambda(\overline{\mathbb{S}}_\mu) = \delta_{\lambda\mu}, \quad |\lambda| \le |\mu| \,.$$

2.7. There is a quite far analogy between central elements $\mathbb{S}_\mu$ and characters of a finite group G considered as elements of the group algebra $\mathbb{C}[G]$ of G. The relationship between quantum immanants and characters of the symmetric groups is especially close (see sections 4 and 5 below). For example, the vanishing and characterization theorems should be considered as an analog of the orthogonality relations for characters of groups.

Suppose π and ρ are two non-equivalent irreducible representations of G and suppose χ^π and χ^ρ are the corresponding irreducible characters. The orthogonality of χ^π and χ^ρ

$$\sum \chi^\rho(g) \cdot \chi^\pi(g^{-1}) = 0$$

can be rewritten as

$$\pi \left(\sum_{g \in G} \chi^{\rho}(g) \cdot g^{-1} \right) = 0 \,,$$

that is as vanishing of some central element in the representation π. In the algebra $\mathcal{U}(\mathfrak{gl}(n))$ there are no elements that vanish in all but one irreducible representations; however the quantum immanants vanish is as many representations as possible. The vanishing and characterization theorems play the same (and, perhaps, more imporatant) role in the combinatorics of s^*-functions as the orthogonality relations in the combinatorics of s-functions (see [OO]).

3. Proof of the main theorem

In this section we shall prove the higher Capelli identities (1.7).

3.1. Consider the following element

$$R(u) = 1 + u \cdot (12) \in \mathbb{k}[S(2)][u] \,.$$

It is called the R-matrix. Normally this R-matrix is denoted by $\check{R}$, but we do not use other R-matrices. The following equation can be verified by direct calculation:

$$(3.1) \qquad R(u-v) \cdot E(u) \otimes E(v) = E(v) \otimes E(u) \cdot R(u-v) \,.$$

This is a version of the famous $RTT = TTR$ equation [RTF]. The equation (1) is equivalent to the commutation relations between the generators E_{ij} of $\mathcal{U}(\mathfrak{gl}(n))$.

3.2. The second key fact we need is the Young orthogonal form [JK]. Put $s_i = (i, i+1) \in S(k)$. Consider the action of s_i in the Young orthogonal basis. Let T be a standard tableau and let $T' = s_i T$ (that is T with i and $i+1$ permuted). Put

$$r = c_T(i+1) - c_T(i) \,.$$

If T' is not a standard tableau then $r = \pm 1$ and

$$s_i \xi_T = \pm \xi_T \,.$$

If T' is a standard tableau then $|r| > 1$ and

$$(3.2) \qquad s_i|_{\mathbb{k}\xi_T + \mathbb{k}\xi_{T'}} = \begin{bmatrix} r^{-1} & (1 - r^{-2})^{1/2} \\ (1 - r^{-2})^{1/2} & -r^{-1} \end{bmatrix} \,.$$

Put $R_i(u) = 1 + u \cdot s_i$ and put

$$R_i(T) = R_i(-r) \,.$$

Clearly,

$$(3.3) \qquad R_i(T)|_{\mathbb{k}\xi_T + \mathbb{k}\xi_{T'}} = \begin{bmatrix} 0 & -r(1 - r^{-2})^{1/2} \\ -r(1 - r^{-2})^{1/2} & 2 \end{bmatrix} \,,$$

if T' is a standard tableau. In any case

$$(3.4) \qquad (R_i(T)\xi_T, \xi_T) = 0 \,.$$

Recall that P_T is the orthogonal projection onto ξ_T. It follows that if T' is standard then

$$(3.5) \qquad R_i(T)P_T = P_{T'} R_i(T)P_T$$

and

$$(3.6) \qquad P_{T'} = (r^2 - 1)^{-1} R_i(T)P_T R_i(T) \,.$$

3.3. Given a standard tableau T put

$$E(T) = E \otimes E(c_T(2)) \otimes \cdots \otimes E(c_T(k))$$

Remark that (1) reads as

$$(3.7) \qquad\qquad R_i(T)E(T) = E(T')R_i(T) \,,$$

where $T' = s_i T$. The third key fact we need is

PROPOSITION.

$$(3.8) \qquad\qquad E(T)P_T = P_T E(T)P_T \,.$$

This proposition is apparently due to I. V. Cherednik [Ch]. (See also [JKMO]).

Proof. First suppose that T is the row tableau T^r of shape μ that is tableau filled in from left to right from top to bottom. For example if $\mu = (3, 2, 1)$ then T^r looks as follows

$$T^r = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{array}$$

Denote by $\mathcal{P}$ the row symmetrizer of T^r

$$\mathcal{P} = \sum_{s \text{ preserves the rows of } T^r} s$$

and by $\mathcal{Q}$ the column antisymmetrizer of T^r

$$\mathcal{Q} = \sum_{s \text{ preserves the columns of } T^r} \operatorname{sgn}(s) \cdot s \,.$$

The product

$$(3.9) \qquad\qquad \mathcal{P}\mathcal{Q} \in \Bbbk[S(k)]$$

is known as the Young symmetrizer corresponindig to the tableau T^r. Denote by W^μ the irreducible $S(k)$-module labeled by μ. The Young symmetrizer acts as zero operator in all irreducible $S(k)$-modules except W^μ. It is clear that

$$\mathcal{P}^2 = \mu! \, \mathcal{P} \,,$$

where $\mu! = \mu_1! \, \mu_2! \, \ldots$. It is well known [JK] that

$$(3.10) \qquad\qquad (\mathcal{P}\mathcal{Q})^2 = H(\mu)\mathcal{P}\mathcal{Q} \,.$$

Hence the element

$$\frac{1}{\,! \, H(\,)} \mathcal{P}\mathcal{Q}\mathcal{P} \in \Bbbk[S(k)]$$

acts as an orthogonal projection in W^μ and as zero operator in other representations of $S(k)$. Again it is well known that the vector ξ_{T^r} is the unique vector in W^μ that is invariant under the action of the row-stabilizer of T^r. Therefore

$$(3.11) \qquad\qquad P_{T^r} = \frac{1}{\mu!\, H(\mu)} \mathcal{P}\mathcal{Q}\mathcal{P}\,.$$

By virtue of (10) we have equality of right ideals

$$(3.12) \qquad\qquad P_{T^r}\, \Bbbk[S(k)] = \mathcal{P}\mathcal{Q}\, \Bbbk[S(k)]\,.$$

Denote the right ideal (12) by I. Consider the annihilator J of I in the semisimple algebra $\Bbbk[S(k)]$

$$J = \{x \in \Bbbk[S(k)] \mid xI = 0\}\,.$$

This is a left ideal in $\Bbbk[S(k)]$. Put

$$M_j = \mu_1 + \cdots + \mu_j\,, \quad j = 1, 2, \ldots\,.$$

In [JKMO] it is shown that J is the left ideal generated by the following elements J_i, $i = 1, \ldots, k-1$

$$J_i = \begin{cases} 1 - (i, i+1), & i \neq M_1, M_2, \ldots \\ (1 + s_{M_{j-1}+1})(1 + 2s_{M_{j-1}+2})\ldots(1 + \mu_j s_{M_j}), & i = M_j\,. \end{cases}$$

Now write

$$E(T^r)P_{T^r} = P_{T^r}E(T^r)P_{T^r} + (1 - P_{T^r})E(T^r)P_{T^r}\,.$$

We are going to show that the second summand vanishes. We have $(1 - P_{T^r})P_{T^r} = 0$ and hence $(1 - P_{T^r}) \in J$. Therefore it suffices to check that

$$J_i E(T^r)P_{T^r} = 0\,, \quad i = 1, 2, \ldots\,.$$

Suppose $i \neq M_1, M_2, \ldots$. Then

$$J_i = R_i(T^r)\,.$$

Therefore by (7)

$$J_i E(T^r)P_{T^r} = E((T^r)')J_i P_{T^r} = 0\,.$$

If $i = M_j$ for some j then we have to apply the same relation (7) several times. Hence (8) is proved for T^r.

Next suppose for some i both T and $T' = s_i T$ are standard tableaux. Show that if (8) holds for T then it also holds for T'. Put

$$r = c_T(i+1) - c_T(i)\,.$$

If T' is a standard tableau then $r \neq \pm 1$. By (6) and (7)

$$\begin{aligned} E(T')P_{T'} &= (r^2 - 1)^{-1}E(T')R_i(T)P_T R_i(T) \\ &= (r^2 - 1)^{-1}R_i(T)E(T)P_T R_i(T)\,. \end{aligned}$$

By assumption this equals to

$$(r^2 - 1)^{-1}R_i(T)P_T E(T)P_T R_i(T)\,.$$

By (5) this expression is stable under multiplication by $P_{T'}$ on the left. Thus (8) is proved for T'. This completes the proof of the proposition. $\square$

3.4. Recall that by definition

$$\mathbb{S}_\mu = \operatorname{tr} E(T) P_T\,,$$

where T is a standard tableau of shape μ.

PROPOSITION. *The quantum immanant $\mathbb{S}_\mu$ is a well defined element of $\mathcal{U}(\mathfrak{gl}(n))$. In other words, the trace*

$$\operatorname{tr} E(T) P_T$$

does not depend on the choice of the standard tableau T of shape μ.

Proof. Show that

$$(3.14) \qquad\qquad \operatorname{tr} E(T) P_T = \operatorname{tr} E(T') P'_T\,,$$

where $T' = s_i T$. Note that if T' is a standard tableau then $R_i(T)$ is invertible. Consider the following chain of equalities

$$
\begin{aligned}
\operatorname{tr} E(T) P_T &= \operatorname{tr} R_i(T) E(T) P_T R_i(T)^{-1} \\
&= \operatorname{tr} E(T') R_i(T) P_T R_i(T)^{-1} & \text{by (7)} \\
&= \operatorname{tr} E(T') P_{T'} R_i(T) P_T R_i(T)^{-1} & \text{by (5)} \\
&= \operatorname{tr} P_{T'} E(T') P_{T'} R_i(T) P_T R_i(T)^{-1} & \text{by (8)} \\
&= \operatorname{tr} E(T') P_{T'} R_i(T) P_T R_i(T)^{-1} P_{T'}\,.
\end{aligned}
$$

It remains to prove that

$$(3.15) \qquad\qquad P_{T'} R_i(T) P_T R_i(T)^{-1} P_{T'} = P_{T'}\,.$$

Clearly (15) holds up to a constant factor. Since the highest terms of (14) agree this constant equals 1. The proposition is proved. $\square$

3.5. Recall that we identify the algebra of right-invariant differential operators on $n \times n$ matrices with $\mathcal{U}(\mathfrak{gl}(n))$.

PROPOSITION.

$$(3.16) \qquad\qquad \mathbb{S}_\mu, \Delta_\mu \in \mathfrak{Z}(\mathfrak{gl}(n))\,.$$

Proof. Denote by g_{ij} and $(g^{-1})_{ij}$ the matrix elements of a matrix $g \in GL(n)$ and its inverse matrix g^{-1}. The following equality is obvious

$$(3.17) \qquad\qquad \sum_k g_{ki}(g^{-1})_{jk} = \delta_{ij}\,.$$

Consider the adjoint action $\operatorname{Ad}(g)$ of g in $\mathfrak{gl}(n)$

$$(3.18) \qquad\qquad \operatorname{Ad}(g) \cdot E_{ij} = \sum g_{ki}(g^{-1})_{jl} E_{kl}\,.$$

Under the adjoint action of g the entries of the matrix $E(u)$ are transformed as follows

$$E(u) \xrightarrow{\mathrm{Ad}(g)} \sum_{i,j} \left(\sum_{k,l} g_{ki}(g^{-1})_{jl} E_{kl} \right) \otimes e_{ij} - u \sum_i 1 \otimes e_{ii} \quad \text{by (18)}$$

$$= \sum_{k,l} (E_{kl} - u\delta_{kl}) \otimes \left(\sum_{i,j} g_{ki}(g^{-1})_{jl} e_{ij} \right) \qquad \text{by (17)}$$

(3.19) $\qquad = g' E(u)(g')^{-1}$

The product (19) is the product of the matrix $E(u)$ with entries in $\mathcal{U}(\mathfrak{gl}(n))$ and two matrices with entries in the ground field $\Bbbk$. Consider the following element of $\mathcal{U}(\mathfrak{gl}(n))$

(3.20) $\qquad\qquad \mathrm{tr}(E(u_1) \otimes \cdots \otimes E(u_k) \cdot s)\,,$

where $u_i \in \Bbbk$ and $s \in S(k)$ are arbitrary. By (19) the adjoint action of g' takes (20) to

$$\mathrm{tr}(g^{\otimes k} E(u_1) \otimes \cdots \otimes E(u_k)(g^{-1})^{\otimes k} \cdot s) = \mathrm{tr}(E(u_1) \otimes \cdots \otimes E(u_k) \cdot s)\,.$$

That is (20) is an element of $\mathfrak{Z}(\mathfrak{gl}(n))$. In particular,

$$\mathbb{S}_\mu \in \mathfrak{Z}(\mathfrak{gl}(n))\,.$$

Under the left action of an element $g \in GL(n)$ the matrices X and D are transformed as follows

$$X \xrightarrow{g} g'X, \quad D \xrightarrow{g} g^{-1}D\,.$$

Therefore the left action of g' takes Δ_μ to

$$\Delta_\mu \xrightarrow{g} \mathrm{tr}\left(g^{\otimes k} \cdot X^{\otimes k} \cdot (D')^{\otimes k} \cdot (g^{\otimes k})^{-1} \cdot \chi^\mu/k! \right)$$
$$= \mathrm{tr}\left(X^{\otimes k} \cdot (D')^{\otimes k} \cdot \chi^\mu/k! \right)$$
$$= \Delta_\mu\,.$$

In the same way the Δ_μ is invariant under the right action of $GL(n)$. Therefore it represent an element of $\mathfrak{Z}(\mathfrak{gl}(n))$. $\quad\square$

3.6. By definition put

(3.21) $\qquad\qquad E(\mu) = \sum_T E(T)P_T\,,$

where the summation is over all standard tableaux T of shape μ. By proposition 3.4 we have

(3.22) $\qquad\qquad \mathbb{S}_\mu = \frac{1}{\dim \mu}\,\mathrm{tr}\, E(\mu)\,.$

PROPOSITION.

$$(3.23) \qquad sE(\mu) = E(\mu)s, \quad \textit{for all } s \in S(k) \,.$$

Proof. We can assume $s \in \Bbbk[S(k)]$. In $\Bbbk[S(k)]$ there is a basis of matrix elements all irreducible representations of $S(k)$ corresponding to the Young orthonormal basis in each representation. If s is a matrix element of a representation ν, $\nu \neq \mu$, then by (8) both LHS and RHS of (23) equal zero. Suppose s is a matrix element of the representation μ. The diagonal matrix elements in the Young basis are proportional to P_T, where T runs over standard tableaux of shape μ. Clearly

$$P_T E(\mu) = E(\mu)P_T = E(T)P_T$$

by (8). Suppose s is a non-diagonal matrix element. We can assume that s takes ξ_T to $\xi_{T'}$, $T' = s_i T$, since such elements are generators. In this case s is proportional to

$$P_{T'} R_i(T) P_T \,.$$

We have

$$
\begin{aligned}
P_{T'} R_i(T) P_T E(\mu) &= P_{T'} R_i(T) P_T E(T) P_T & \text{by (8)} \\
&= P_{T'} R_i(T) E(T) P_T & \text{by (8)} \\
&= P_{T'} E(T') R_i(T) P_T & \text{by (7)} \\
&= P_{T'} E(T') P_{T'} R_i(T) P_T & \text{by (5)} \\
&= E(T') P_{T'} R_i(T) P_T & \text{by (8)} \\
&= E(\mu) P_{T'} R_i(T) P_T \,,
\end{aligned}
$$

as desired. $\quad\square$

Suppose we have a sequence of indices $i_1 \leq i_2 \leq \cdots \leq i_k$ or $i_1 \geq i_2 \geq \cdots \geq i_k$. Suppose that exactly ι_1 first indices are equal, exactly ι_2 next indices are equal and so on. Then the stabilizer in $S(k)$ of the sequence $(i_1, \ldots, i_k)$ is isomorphic to

$$S(\iota) = S(\iota_1) \times S(\iota_2) \times \ldots \,.$$

We have

$$|S(\iota)| = \iota! = \iota_1! \iota_2! \ldots \,.$$

For example, if all i_j are distinct then $\iota = (1^k)$.

COROLLARY.

$$(3.24) \quad \mathbb{S}_\mu = \sum_{i_1 \geq \cdots \geq i_k} 1/\iota! \sum_T \sum_{s \in S(k)} (s \cdot \xi_T, \xi_T) \, E_{i_1, i_{s(1)}} (E_{i_2, i_{s(2)}} - c_T(2)\delta_{i_2, i_{s(2)}}) \ldots \,.$$

Proof. By (23) all $k!/\iota!$ summands corresponding to different rearrangements of $\{i_1, \ldots, i_k\}$ make the same contribution to the sum (22). $\quad\square$

In the same way we can write

$$(3.24') \quad \mathbb{S}_\mu = \sum_{i_1 \leq \cdots \leq i_k} 1/\iota! \sum_T \sum_{s \in S(k)} (s \cdot \xi_T, \xi_T) \, E_{i_1, i_{s(1)}} (E_{i_2, i_{s(2)}} - c_T(2)\delta_{i_2, i_{s(2)}}) \ldots \,.$$

The formula (24$'$) turns into the original definition (1.1) of the Capelli element if

3.7. Now we can calculate the eigenvalues of the quantum immanants.

By $\mathrm{RTab}(\mu, n)$ denote the set of reverse column strict tableau T of shape μ with entries in $\{1, \ldots, n\}$. By definition $T \in \mathrm{RTab}(\mu, n)$ if entries of T weakly decrease along the rows and strictly decrease along the columns. By definition, put

$$(3.25) \qquad \Sigma_\mu(x_1, \ldots, x_n) = \sum_{T \in \mathrm{RTab}(\mu, n)} \prod_{\alpha \in \mu} (x_{T(\alpha)} - c(\alpha)),$$

where the product is over all cells α of μ and $c(\alpha)$ denotes the content of the cell α. For factorial Schur polynomials sums analogous to (25) were considered by Biedenharn and Louck [BL], Chen and Louck [CL], Goulden and Hamel [GH], Macdonald [M2] and others.

Since the content of the cell $(1, 1) \in \mu$ equals 0 the sum (25) is stable in the following sense

$$(3.26) \qquad \Sigma_\mu(x_1, \ldots, x_n, 0) = \Sigma_\mu(x_1, \ldots, x_n).$$

Let λ be a partition. By (26) the number $\Sigma_\mu(\lambda)$ is well defined.

PROPOSITION.

$$(3.27) \qquad \pi_\lambda(\mathbb{S}_\mu) = \Sigma_\mu(\lambda).$$

Proof. Apply (24) to the highest vector. Since the highest vector is annihilated by all E_{ij} with $i < j$ we get

$$(3.28) \qquad \pi_\lambda(\mathbb{S}_\mu) = \sum_{i_1 \geq \cdots \geq i_k} 1/\iota! \sum_{T} \sum_{s \in S(\iota)} (s \cdot \xi_T, \xi_T) \, \lambda_{i_1}(\lambda_{i_2} - c_T(2)) \ldots .$$

Here the summation in s is over the stabilizer $S(\iota) \subset S(k)$ of $i_1, \ldots, i_k$.

Given a standard tableau T denote by $\widetilde{T} = i(T)$ the tableau obtained by replacing each number j in T by i_j. The entries in rows and columns of $i(T)$ weakly decrease. We have

$$(3.29) \quad \pi_\lambda(\mathbb{S}_\mu) = \sum_{i_1 \geq \cdots \geq i_k} 1/\iota! \sum_{\widetilde{T}} \lambda_{i_1}(\lambda_{i_2} - c_T(2)) \ldots \left(\sum_{s \in S(\iota)} \sum_{T \in i^{-1}(\widetilde{T})} (s \cdot \xi_T, \xi_T) \right).$$

Here $\widetilde{T}$ runs over all tableaux with entries $i_1, \ldots, i_k$ and weakly decreasing rows and columns. Next we show that

$$(3.30) \qquad \left(\sum_{s \in S(\iota)} \sum_{T \in i^{-1}(\widetilde{T})} (s \cdot \xi_T, \xi_T) \right) = \begin{cases} \iota!, & \text{if } \widetilde{T} \text{ is column strict} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the diagram μ as disjoint union of skew diagrams $\mu_1, \mu_2, \ldots$ (which depend

the diagram μ_2 consists of next ι_2 cells of $\widetilde{T}$ and so on. Then it is easy to see that the LHS of (30) equals

$$(3.31) \qquad \prod_m \sum_{s \in S(\iota_m)} \chi^{\mu_m}(s) \, .$$

It is an elementary fact from the representation theory of the symmetric group that for any skew diagram η

$$(3.32) \qquad \sum_{s \in S(|\eta|)} \chi^\eta(s) = \begin{cases} |\eta|!, & \text{if } \eta \text{ is a horizontal strip,} \\ 0, & \text{otherwise} \, . \end{cases}$$

This proves (30) and therefore

$$\pi_\lambda(\mathbb{S}_\mu) = \sum_{i_1 \geq \cdots \geq i_k} \sum_{\widetilde{T} \in \mathrm{RTab}(\mu,n)} \prod_{\alpha \in \mu} (\lambda_{\widetilde{T}(\alpha)} - c(\alpha)) \, .$$

Finally observe that the summation over $i_1, \ldots, i_k$ can be eliminated if we allow $\widetilde{T}$ to range over all inverse column strict tableaux of shape μ and entries $1, \ldots, n$. Hence

$$(3.33) \qquad \pi_\lambda(\mathbb{S}_\mu) = \sum_{\widetilde{T} \in \mathrm{RTab}(\mu,n)} \prod_{\alpha \in \mu} (\lambda_{\widetilde{T}(\alpha)} - c(\alpha)) \, .$$

Clearly this is the desired formula. $\square$

3.8. In this paragraph we prove the following

PROPOSITION.

$$(3.34) \qquad \pi_\lambda(\mathbb{S}_\mu) = s_\mu^*(\lambda) \, .$$

By virtue of (27) this proposition is equivalent to the following *combinatorial* formula for s^*-functions.

$$(3.35) \qquad s_\mu^*(x_1, x_2, \ldots) = \Sigma_\mu(x_1, x_2, \ldots) \, .$$

This formula is eqivalent to the analogous formula for factorial Schur functions (see [CL],[GG] and [M2]). A proof of (35) is given also in [OO]. Below we give one more proof of (35) based on the characterization theorem.

Proof. The number $\Sigma_\mu(\lambda)$ equals by (27) to the eigenvalue of an element of $\mathfrak{Z}(\mathfrak{gl}(n))$ and hence $\Sigma_\mu \in \Lambda^*(n)$.

Show that $\Sigma_\mu(\lambda) = 0$ unless $\mu \in \lambda$. Moreover, show that

$$(3.36) \qquad \prod_{\alpha \in \mu} (\lambda_{T(\alpha)} - c(\alpha)) = 0$$

for all $T \in \mathrm{RTab}(\mu, n)$ unless $\mu \in \lambda$. Denote the LHS of (36) by Π_T. Put $\lambda_{(i,j)} = \lambda_{T(i,j)}$. Since $T \in \mathrm{RTab}(\mu, n)$ we have

(3.37)

If $\Pi_T(\lambda) \neq 0$ then

(3.38) $$\lambda_{(1,1)} \neq 0, \quad \lambda_{(1,2)} \neq 1, \quad \lambda_{(1,3)} \neq 2, \ldots$$

By (37) and (38) we have

(3.39) $$\lambda_{(1,1)} \geq 1, \quad \lambda_{(1,2)} \geq 2, \quad \lambda_{(1,3)} \geq 3, \ldots$$

Again since $T \in \mathrm{RTab}(\mu, n)$ we have

(3.40) $$T(1, i) < T(2, i) < \cdots < T(\mu'_i, i)$$

and we have also

(3.41) $$i \leq \lambda_{(1,i)} \leq \lambda_{(2,i)} \leq \cdots \leq \lambda_{(\mu'_i, i)}$$

for all i. Observe that (40) and (41) yield $\lambda'_i \geq \mu'_i$. Thus $\Pi_T(\lambda) \neq 0$ implies $\mu \subset \lambda$. By the characterization theorem Σ_μ equals s^*_μ up to a constant factor. In order to see that this factor equals 1 we can either compare the highest terms of s^*_μ and Σ_μ or calculate explicitly the unique non-vanishing summand in Σ_μ. $\square$

It is interesting to look at shifted analogs of elementary and complete homogeneous functions. By (35) we have

$$s^*_{(1^k)}(x) = \sum_{i_1 < \cdots < i_k} (x_{i_1} + k - 1) \ldots (x_{i_{k-1}} + 1) x_{i_k}$$

$$s^*_{(k)}(x) = \sum_{i_1 \leq \cdots \leq i_k} (x_{i_1} - k + 1) \ldots (x_{i_{k-1}} - 1) x_{i_k} .$$

3.9. Now we can complete the proof of the theorem. Consider the difference

(3.42) $$\mathbb{S}_\mu - \Delta_\mu .$$

As explained in paragraph 1.4 this is an element of degree $< |\mu|$. Next (42) is a central element. Prove that it vanishes in all representations π_λ such that $|\lambda| < |\mu|$. We have proved in the previous paragraph that $\pi_\lambda(\mathbb{S}_\mu) = 0$ for such λ. The differential operator vanishes also. Indeed all irreducible $GL(n)$-submodules of $\Bbbk[\mathrm{M}(n)]$ with highest weight λ consist of polynomials of degree $|\lambda| < |\mu|$. Such polynomials are clearly annihilated by the operator Δ_μ. Thus by the characterization theorem (42) equals zero. This concludes the proof of the theorem.

3.10. The quantum immanant $\mathbb{S}_\mu$ can be expressed in terms of the Young symmetrizer (9). We keep the notations of paragraph 3.3. The element

$$H(\mu)^{-1} \mathcal{P} \mathcal{Q}$$

is an idempotent proportional to the Young symmetrizer. Consider the following element of $\mathcal{U}(\mathfrak{gl}(n))$

(3.43) $$H(\mu)^{-1} \operatorname{tr} E(T^r) \mathcal{P} \mathcal{Q} .$$

We have

$$\begin{aligned}
H(\mu)^{-1} \operatorname{tr} E(T^r) \mathcal{P} \mathcal{Q} &= H(\mu)^{-1} \operatorname{tr} E(T^r) P_{T^r} \mathcal{P} \mathcal{Q} & \text{by (11)} \\
&= H(\mu)^{-1} \operatorname{tr} P_{T^r} E(T^r) \mathcal{P} \mathcal{Q} & \text{by (8)} \\
&= H(\mu)^{-1} \operatorname{tr} E(T^r) \mathcal{P} \mathcal{Q} P_{T^r} \\
&= \operatorname{tr} E(T^r) P_{T^r} & \text{by (11)} \\
&= \mathbb{S}_\mu .
\end{aligned}$$

4.Higher Capelli identities for Schur-Weyl duality.

4.1. Consider the space of tensors $(\Bbbk^n)^{\otimes K}$. It is a multiplicity free $GL(n) \times S(K)$-module; so we can look for Capelli identities (in the sence of [HU]) for it.

Suppose $k \leq K$ and $|\mu| = k$. Embed $S(k)$ in $S(K)$. Denote by $\mathrm{Ind}\,\chi^\mu$ the induced character of $S(K)$. By the Frobenius formula

$$(4.1) \qquad \mathrm{Ind}\,\chi^\mu = \sum_{t \in S(K)/S(k)} t \cdot \chi^\mu \cdot t^{-1},$$

in other words $\mathrm{Ind}\,\chi^\mu$ is proportional to the averaging of $\chi^\mu \in \Bbbk[S(k)]$ over the group $S(K)$.

Let τ denote the representation of the group $GL(n)$ in the space $(\Bbbk^n)^{\otimes K}$ and let σ denote the representation of the group $S(K)$ in the same space.

Theorem.

$$(4.2) \qquad \tau(\mathbb{S}_\mu) = \sigma(\mathrm{Ind}\,\chi^\mu \,/\, (K-k)!)\,.$$

Proof. Denote by $\mathrm{M}(n,K)$ the space of rectangular $n \times K$ matrices. Let $\{e_i\}$, $i = 1, \ldots, n$ be the standard basis of $\Bbbk^n$. Embed $(\Bbbk^n)^{\otimes K}$ in $\Bbbk[\mathrm{M}(n,K)]$ as follows

$$(4.3) \qquad e_{i_1} \otimes \cdots \otimes e_{i_K} \to x_{i_1 1} \ldots x_{i_K K}\,.$$

This embedding is $GL(n)$-equivariant. By (1.7) the operator $\tau(\mathbb{S}_\mu)$ becomes Δ_μ. Consider the action of the group $S(K)$

$$s \cdot x_{i_1 1} \ldots x_{i_K K} = x_{i_1 s^{-1}(1)} \ldots x_{i_K s^{-1}(K)}\,.$$

By its very definition the operator Δ_μ acts as follows

$$x_{i_1 1} \ldots x_{i_K K} \xrightarrow{\Delta_\mu} \sum_{t \in S(K)/(S(k) \times S(K-k))} \sum_{s \in S(k)} \chi^\mu(s)\,(tst^{-1}) \cdot x_{i_1 1} \ldots x_{i_K K}\,.$$

Thus Δ_μ acts in the same way as $\mathrm{Ind}\,\chi^\mu \,/\, (K-k)!$ $\square$

Another approach to Capelli-type identities for Schur-Weyl duality was developed in [KO]

5. Further properties of quantum immanants.

The results of this sections are from [OO] (only the proofs differ). This results are based on (2.2); that is they are essentially properties of s^*-functions.

The proofs below use higher Capelli identities. One can take a short-cut and deduce all propositions directly from the characterization theorem. Such proofs can be found in [OO].

5.1. We have considered $\mathbb{S}_\mu \in \mathfrak{Z}(\mathfrak{gl}(n))$ where n was a fixed number. Now let n vary.

Suppose $N > n \geq \ell(\mu)$. Consider the composition of the two maps

$$(5.1) \qquad \mathfrak{Z}(\mathfrak{gl}(n)) \to \mathcal{U}(\mathfrak{gl}(N)) \to \mathfrak{Z}(\mathfrak{gl}(N))\,,$$

where the first arrow is the natural inclusion and the second one is the $\mathfrak{gl}(N)$-invariant projection. If $\Bbbk = \mathbb{C}$ then this composition is the averaging over the group $U(N)$. We call this map the *averaging* map. We denote the averaging of $\xi \in \mathfrak{Z}(\mathfrak{gl}(n))$ by $\langle \xi \rangle_N$.

In order to avoid confusion denote by by $\overline{\mathbb{S}}_{\mu|n}$ the normalized quantum μ-immanant in $\mathfrak{Z}(\mathfrak{gl}(n))$. We call the following property the *coherence* of quantum

PROPOSITION [OO].

$$(5.2) \qquad \left\langle \overline{\mathbb{S}}_{\mu|n} \right\rangle_N = \overline{\mathbb{S}}_{\mu|N}$$

Proof. Identify $\mathcal{U}(\mathfrak{gl}(n))$ with distributions supported at $1 \in \mathrm{M}(n)$. By $s_\mu(D)$ denote the polynomial (1.9) in variables ∂_{ij}.

LEMMA 1.

$$(5.3) \qquad (\mathbb{S}_\mu, \phi) = \left[s_\mu(D) \cdot \phi \right](1).$$

Proof of Lemma. The higher Capelli identity (1.7) asserts that

$$(\mathbb{S}_\mu, \phi) = \left[\mathrm{tr}(X^{\otimes k} \cdot (D')^{\otimes k} \cdot \chi^\mu / k!) \cdot \phi \right](1)$$
$$(5.4) \qquad\qquad = \left[\mathrm{tr}((D')^{\otimes k} \cdot \chi^\mu / k!) \cdot \phi \right](1),$$

where we used the fact that $x_{ij}(1) = \delta_{ij}$. By (1.9) we have

$$(\mathbb{S}_\mu, \phi) = \left[s_\mu(D') \cdot \phi \right](1).$$

Since $\chi^\mu(s) = \chi^\mu(s^{-1})$ for all $s \in S(k)$ we have

$$s_\mu(D') = s_\mu(D). \quad \square$$

As in (1) consider the composition of the following inclusion and projection

$$(5.5) \qquad S(\mathfrak{gl}(n))^{GL(n)} \to S(\mathfrak{gl}(N)) \to S(\mathfrak{gl}(N))^{GL(N)},$$

where $S(\mathfrak{gl}(n))$ is the symmetric algebra of $\mathfrak{gl}(n)$ and $S(\mathfrak{gl}(n))^{GL(n)}$ denotes the invariants of the adjoint action of $GL(n)$. We call this map the averaging map also. Using the invariant scalar product in $\mathfrak{gl}(n)$

$$(5.6) \qquad (A, B) = \mathrm{tr}\, AB, \quad A, B \in \mathfrak{gl}(n),$$

we construct a similar averaging map

$$(5.7) \qquad \Bbbk[\mathrm{M}(n)]^{GL(n)} \to \Bbbk[\mathrm{M}(N)] \to \Bbbk[\mathrm{M}(N)]^{GL(N)}.$$

Again to avoid confusion denote $s_{\mu|n}(X)$ the polynomial (1.9) in matrix elements of a $n \times n$ matrix X.

LEMMA 2.

$$(5.8) \qquad \left\langle s_{\mu|n}(X) \right\rangle_N = \frac{(n \upharpoonright \mu)}{(N \upharpoonright \mu)}\, s_{\mu|N}(X) \qquad \in \Bbbk[\mathrm{M}(N)]^{GL(N)},$$

$$(5.9) \qquad \left\langle s_{\mu|n}(D) \right\rangle_N = \frac{(n \upharpoonright \mu)}{(N \upharpoonright \mu)}\, s_{\mu|N}(D) \qquad \in S(\mathfrak{gl}(N))^{GL(N)}.$$

Proof of Lemma. Recall that $s_\mu(X)$ equals the trace of $X = (x_{ij})$ in the irreducible $GL(n)$-module with highest weight μ. Consider the matrix element f_λ corresponding to the highest vector

$$f_\lambda = \prod_{i=1}^{\ell(\mu)} \det \begin{bmatrix} x_{11} & \cdots & x_{1i} \\ \vdots & & \vdots \\ x_{i1} & \cdots & x_{ii} \end{bmatrix}^{\mu_i - \mu_{i+1}} \in \mathbb{k}[\mathrm{M}(\ell(\mu))] \,.$$

The averaging of a matrix element equals the trace:

$$(5.10) \qquad \langle f_\lambda \rangle_n = \frac{1}{\dim_{GL(n)} \lambda} \, s_{\mu|n}(X) \,,$$

where $\dim_{GL(n)} \lambda$ denotes the dimension of the irreducible $GL(n)$-module with highest weight λ. Hence

$$\begin{aligned}
\langle s_{\mu|n}(X) \rangle_N &= \dim_{GL(n)} \lambda \, \langle f_\lambda \rangle_N \\
(5.11) \qquad &= \frac{\dim_{GL(n)} \lambda}{\dim_{GL(N)} \lambda} \, s_{\mu|N}(X) \,.
\end{aligned}$$

Now (8) follows from (2.16). The claim (9) follows from (8) and the formula (6) for the invariant scalar product. □

The proposition follows immediately from (2),(9) and the definition of $\overline{\mathbb{S}}_\mu$. □

5.2. There is a map in the inverse direction

$$\mathfrak{Z}(\mathfrak{gl}(N)) \to \mathfrak{Z}(\mathfrak{gl}(n)) \,,$$

which is the restriction of invariant diffential operators on $\mathrm{M}(N)$ to the invariant subspace $\mathbb{k}[\mathrm{M}(n)]$. This map was studied by Olshanski. It plays the central role in [Ol1]. It follows from the very definition of the operator Δ_μ that

$$(5.12) \qquad \mathbb{S}_{\mu|N} \xrightarrow{\text{restriction}} \mathbb{S}_{\mu|n}$$

provided $n \geq \ell(\mu)$. On the level of eigenvalues (12) is equivalent to the stability (2.4) of s^*-functions.

5.3. Suppose $|\lambda| = K$. By $\dim \lambda/\mu$ denote the dimension of the skew Young diagram λ/μ. This number equals

$$(5.14) \qquad \dim \lambda/\mu = \left\langle \mathrm{Res}\, \chi^\lambda, \chi^\mu \right\rangle_{S(k)} \,,$$

where $\mathrm{Res}\, \chi^\lambda$ is the restriction of χ^λ to $S(k)$ and $\langle \cdot, \cdot \rangle$ is the standard scalar product of functions on $S(k)$

$$\langle \phi, \psi \rangle_{S(k)} = (1/k!) \sum_{s \in S(k)} \phi(s) \psi(s^{-1}) \,.$$

Equivalently $\dim \lambda/\mu$ is equal to the number of paths from μ to λ in the Young graph. Recall that the Young graph is the oriented graph whose vertices are partitions and two partitions μ and ν are connected by an edge (write $\mu \nearrow \nu$) if ν/μ is

PROPOSITION [OO].

$$(5.15) \qquad \frac{\dim \lambda/\mu}{\dim \lambda} = \frac{s_\mu^*(\lambda)}{(|\lambda| \downarrow |\mu|)} \,.$$

Proof. Compare the eigenvalues of both sides of (4.2) in the irreducible submodule corresponding to λ. By definition of $\mathbb{S}_\mu$ its eigenvalue equals $s_\mu^*(\lambda)$. Calculate the eigenvalue of $\operatorname{Ind} \chi^\mu$. Its trace equals

$$\begin{aligned}
\chi^\lambda(\operatorname{Ind} \chi^\mu) &= K! \, \langle \chi^\lambda, \operatorname{Ind} \chi^\mu \rangle_{S(K)} \\
&= K! \, \langle \operatorname{Res} \chi^\lambda, \chi^\mu \rangle_{S(k)} \quad \text{by the Frobenius reciprocity} \\
(5.16) \qquad &= K! \dim \lambda/\mu \,.
\end{aligned}$$

Hence the eigenvalue of the left hand side of (4.2) equals

$$\frac{K!}{(K-k)!} \frac{\dim \lambda/\mu}{\dim \lambda} = (|\lambda| \downarrow |\mu|) \frac{\dim \lambda/\mu}{\dim \lambda} \,.$$

This yields (15). $\square$

5.4. The main application of the formulas (2) and (15) is the explicit solution of the two following problems:

(1) given an element $s \in \Bbbk[S(k)]$ and a character χ^λ of the group $S(K)$, $K > k$ find
$$\frac{\chi^\lambda(s)}{\dim \lambda} \,,$$

(2) given an element $\xi \in \mathcal{U}(\mathfrak{gl}(n))$ and a representation π_λ of the group $GL(N)$, $N > n$ find
$$\frac{\operatorname{tr} \pi_\lambda(\xi)}{\dim \lambda} = \pi_\lambda(\langle \xi \rangle_N) \,,$$

where $\langle \xi \rangle_N$ is the averaging of ξ.

Indeed, it is clear that s and ξ can be assumed to be central. In the center of $\Bbbk[S(k)]$ and $\mathcal{U}(\mathfrak{gl}(n))$ there is the basis of irreducible characters and quantum immanants respectively. Finally observe that the problems are linear in s and ξ respectively.

These problems play the key role in the ergodic method of A. M. Vershik and S. V. Kerov [VK]. In fact the understanding of the papers [VK] was the original aim of G. Olshanski and me. These problems are also discussed in [KO].

The solution of similar problems of other classical groups will be given in a forthcoming paper by G. Olshanski and me.

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